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Oriented unicyclic graphs with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies☆

Jianming Zhu *

Department of Mathematics, Tongji University, Shanghai 200092, China

Business Information Management School, Shanghai Institute of Foreign Trade, Shanghai 201620, China

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ABSTRACT

Let G^σ be an oriented graph obtained by assigning an orientation σ to the edge set of a simple undirected graph G such that G^σ becomes a directed graph. Let $S(G^\sigma)$ be the skew adjacency matrix of G^σ. The skew energy of G^σ is defined as the sum of the absolute values of all eigenvalues of $S(G^\sigma)$. In this paper, we provide a new method to compare the skew energies of two oriented graphs whose skew characteristic polynomials satisfy a given recurrence relation and determine the oriented unicyclic graphs of order n with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies for $n \geqslant 31$.

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1. Introduction

Let G be a simple undirected graph. The energy of G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix (see [1–3]). In theoretical chemistry, the energy of a given molecular graph is related to the total π-electron energy of the molecule represented by that graph. Consequently the graph energy has some specific chemistry interests and has been extensively studied [1–3,12–27].

One of the graph classes that has been quite thoroughly studied is the class of all unicyclic graphs [12–24], i.e., connected graphs with one unique cycle. A number of results concerning the extremal

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* Address: Department of Mathematics, Tongji University, Shanghai 200092, China.
E-mail address: zhujianming2005@163.com

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energies of various families of unicyclic graphs have been obtained. Hou [20] characterized the unique unicyclic graph with the minimal energy. In [16], Huo et al. determined the unique unicyclic graph with the maximal energy. Recently, Andriantiana and Wagner [15] further determined the unique unicyclic graphs with the second maximal energy.

Let G^σ be an oriented graph obtained by assigning an orientation σ to the edge set of a simple undirected graph G such that G^σ becomes a directed graph. The skew adjacency matrix $S(G^\sigma) = (s_{ij})$ of G^σ is a real skew symmetric matrix, where $s_{ij} = 1$ and $s_{ji} = -1$ if ij is an arc of G^σ with tail i and head j, otherwise $s_{ij} = s_{ji} = 0$. The skew spectrum $Sp(G^\sigma)$ of G^σ is defined as the spectrum of $S(G^\sigma)$. Note that $Sp(G^\sigma)$ consists of only purely imaginary eigenvalues because $S(G^\sigma)$ is real skew symmetric. Shader and So [7] studied the skew spectrum of oriented graphs and obtained some results.

There are various generalizations of graph energy, such as Laplacian energy, incidence energy and energy of a polynomial [4,5]. Recently, the skew energy of an oriented graph G^σ, denoted by $E_s(G^\sigma)$, is defined as the sum of the absolute values of all eigenvalues of $S(G^\sigma)$ (see [6]), that is,

$$E_s(G^\sigma) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the all eigenvalues of $S(G^\sigma)$. Recently, the skew energy of oriented graphs has been studied in [8–11].

The characteristic polynomial $det(xI - S(G^\sigma))$ of the skew adjacency matrix $S(G^\sigma)$ of an oriented graph G^σ is also called the skew characteristic polynomial of G^σ, written as $\phi(G^\sigma, x) = \sum_{i=0}^{n} a_i(G^\sigma)x^{n-i}$. Since $S(G^\sigma)$ is a real skew symmetric matrix, we have $a_{2i}(G^\sigma) \geqslant 0$ and $a_{2i+1}(G^\sigma) = 0$ for all $0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$ (see [6]). Then we have

$$\phi(G^\sigma, x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i}(G^\sigma)x^{n-2i}. \tag{1}$$

Using these coefficients of $\phi(G^\sigma, x)$, the skew energy $E_s(G^\sigma)$ of an oriented graph G^σ of order n can be expressed by the following integral formula [11]:

$$E_s(G^\sigma) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left[\sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i}(G^\sigma)x^{2i} \right] dx. \tag{2}$$

Note that $a_0(G^\sigma) = 1$ and $a_2(G^\sigma)$ equals to the number of the edges of G. It follows that $E_s(G^\sigma)$ is a strictly monotonically increasing function of those numbers $a_{2i}(G^\sigma)(i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor)$ for any oriented graphs. This in turn provides a way of comparing the skew energies of a pair of oriented graphs. That is to say, the method of the quasi-order relation defined by Gutman and Polansky [2] on graph energy can be generalized to the skew energy of oriented graph as follows:

Definition 1.1. Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be two oriented graphs of order n. If $a_{2i}(G_1^{\sigma_1}) \leqslant a_{2i}(G_2^{\sigma_2})$ for all i with $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, then we write $G_1^{\sigma_1} \preceq G_2^{\sigma_2}$.

Furthermore, if $G_1^{\sigma_1} \preceq G_2^{\sigma_2}$ and there exists at least one index j such that $a_{2j}(G_1^{\sigma_1}) < a_{2j}(G_2^{\sigma_2})$, then we write that $G_1^{\sigma_1} \prec G_2^{\sigma_2}$. If $a_{2i}(G_1^{\sigma_1}) = a_{2i}(G_2^{\sigma_2})$ for all i, we write $G_1^{\sigma_1} \sim G_2^{\sigma_2}$. According to the integral formula (2), we have for two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ of order n that

$G_1^{\sigma_1} \preceq G_2^{\sigma_2} \Rightarrow E_s(G_1^{\sigma_1}) \leqslant E_s(G_2^{\sigma_2})$;
$G_1^{\sigma_1} \prec G_2^{\sigma_2} \Rightarrow E_s(G_1^{\sigma_1}) < E_s(G_2^{\sigma_2})$.

Let G^σ be an oriented graph and C be an even cycle of G. We say C is evenly oriented relative to the orientation σ if it has an even number of edges oriented in the same direction (clockwise or anti-clockwise direction). Otherwise C is said to be oddly oriented (see Fig. 1).

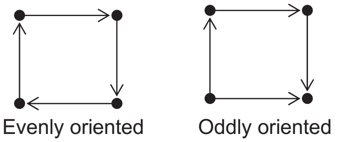

Fig. 1. Two orientations of C_4.

A linear subgraph L of G is a disjoint union of some edges and some cycles in G. We call a linear subgraph L of G evenly linear if L contains no odd cycle. We denote by $\mathcal{EL}_{2i}(G)$ the set of all evenly linear subgraphs of G with $2i$ vertices. For an evenly linear subgraph $L \in \mathcal{EL}_{2i}(G)$, we denote by $p_e(L)$ the number of evenly oriented cycles in L relative to the orientation σ.

About the coefficients $a_{2i}(G^\sigma)$ of the skew characteristic polynomial of an oriented graph G^σ, Hou and Lei [10] obtained the following result.

Lemma 1.1 [10]. *Let G^σ be an oriented graph. Then*

$$a_{2i}(G^\sigma) = \sum_{L \in \mathcal{EL}_{2i}(G)} (-1)^{p_e(L)} 2^{c(L)},$$

where $p_e(L)$ is the number of evenly oriented cycles of L and $c(L)$ is the number of even cycles of L, respectively.

In this paper, we denote by $G(n, l)$ the set of all unicyclic graphs of order n with the unique cycle C_l. A $k-matching$ is a disjoint union of k edges in G. The number of $k-matching$ is denoted by $m(G, k)$. We agree that $m(G, 0) = 1$ and $m(G, k) = 0 (k < 0)$.

About the coefficients $a_{2i}(G^\sigma)$ of an oriented unicyclic graph, Hou et al. [11] obtained the following lemma.

Lemma 1.2 [11]. *Let $G \in G(n, l)$ and σ be an orientation of G. Then we have:*

(1) *If l is odd, then $a_{2i}(G^\sigma) = m(G, i)$;*
(2) *If l is even and C_l is oddly oriented, then $a_{2i}(G^\sigma) = m(G, i) + 2m\left(G - C_l, i - \frac{l}{2}\right)$;*
(3) *If l is even and C_l is evenly oriented, then $a_{2i}(G^\sigma) = m(G, i) - 2m\left(G - C_l, i - \frac{l}{2}\right)$.*

Let $G \in G(n, l)$ and Γ be the set of all orientations of G.

Suppose that l is odd. By Lemma 1.1, $\forall \sigma_1, \sigma_2 \in \Gamma$, we have $G^{\sigma_1} \sim G^{\sigma_2}$. For convenience, $\forall \sigma \in \Gamma$, we write G^* as G^σ when l is odd.

Suppose that l is even and $\Gamma_1 = \{\sigma | \sigma \in \Gamma$ and C_l is oddly oriented relative to $\sigma\}$. By Lemma 1.1, $\forall \sigma_1, \sigma_2 \in \Gamma_1$, we have $G^{\sigma_1} \sim G^{\sigma_2}$. For convenience, $\forall \sigma \in \Gamma_1$, we write G^+ as G^σ.

Suppose that l is even and $\Gamma_2 = \{\sigma | \sigma \in \Gamma$ and C_l is evenly oriented relative to $\sigma\}$. By Lemma 1.2, $\forall \sigma_1, \sigma_2 \in \Gamma_2$, we have $G^{\sigma_1} \sim G^{\sigma_2}$. For convenience, $\forall \sigma \in \Gamma_2$, we write G^- as G^σ.

Let S_5^4 be the graph obtained by attaching a pendent vertex to a vertex of C_4. We denote by $U_4(a, b)$ the graph obtained by attaching two pendent paths of length a and b to the unique pendent vertex of

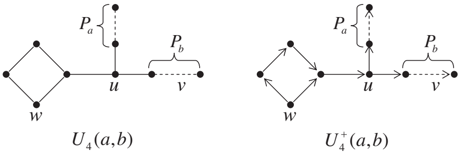

Fig. 2. The graph $U_4(a, b)$ and an orientation of $U_4(a, b)$.

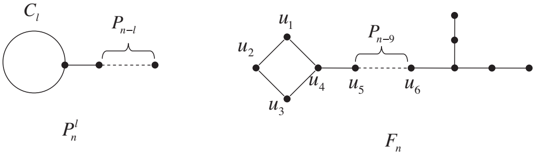

Fig. 3. Graphs P_n^l and F_n.

S_5^4 (see Fig. 2). When the cycle C_4 of $U_4(a, b)$ is oddly oriented relative to an orientation σ, we write $U_4^+(a, b)$ as $[U_4(a, b)]^\sigma$.

Let a,b be nonnegative integers and $\mathcal{A}_n^+ = \{U_4^+(a, b) | 0 \leqslant a \leqslant b, a + b = n - 5, a \neq 1, 3, 5\}$. In [11], Hou et al. shown that the oriented unicyclic graphs of order n with the maximal skew energy are $U_4^+(0, n-5)$. In this paper, we further show that the oriented unicyclic graphs of order $n \geqslant 31$ with the first $\left\lfloor \frac{n-9}{2} \right\rfloor$ largest skew energies are the oriented unicyclic graphs in $\mathcal{A}_n^+$.

The rest of this paper is organized as follows: In section 2, we investigate the skew energies of the oriented unicyclic graphs of order n where l is odd. Then section 3 is devoted to study the skew energies of the oriented unicyclic graphs of order n where l is even. In section 4, we present a new method of comparing the skew energies of two oriented graphs of order n whose skew characteristic polynomials satisfy a given recurrence relation. The oriented unicyclic graphs of order $n \geqslant 31$ with the first $\left\lfloor \frac{n-9}{2} \right\rfloor$ largest skew energies are determined in section 5.

2. *l* is odd

Let $G \in G(n, l)$ and C_l be the unique cycle of G. In this section, we always assume that l is odd.

Let P_n^l be the unicyclic graph of order n obtained by attaching a pendent path of length $n - l$ $(n > l)$ to a vertex of a cycle C_l. We denote by F_n the graph obtained by attaching two pendent paths of length 2 to the unique pendent vertex of the graph P_{n-4}^4 (see Fig. 3). In this section, we will show that $G^\sigma \prec F_n^+$ in Lemmas 2.2 and 2.3.

The following lemma is the main method used to compare the matching numbers of two graphs in this paper.

Lemma 2.1 [4]. *Let G be a graph of order n and uv be an edge of G. Then*

$$m(G, k) = m(G - uv, k) + m(G - u - v, k - 1) \qquad \left(1 \leqslant k \leqslant \left\lfloor \frac{n}{2} \right\rfloor\right).$$

The following result can be derived from Lemma 1.2.

Lemma 2.2. *Let $G \in G(n, l)$ and σ be any orientation of G. Let l be odd. Then we have:*

(1) *If n is even, then $G^\sigma \prec C_n^+$;*
(2) *If n is odd, then $G^\sigma \preceq C_n^*$.*

Proof. (1) Since l is odd, by Lemma 1.2 we have $a_{2i}(G^\sigma) = m(G, i)$. We consider the following two cases.

Case 1: $2i = n$.
By Lemma 1.2 we have, $a_n(G^\sigma) = m(G, \frac{n}{2}) \leqslant 2$ and $a_n(C_n^+) = m\left(C_n, \frac{n}{2}\right) + 2 = 4$. Then $a_n(G^\sigma) < a_n(C_n^+)$.

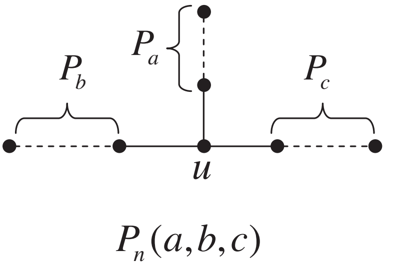

Fig. 4. The graph $P_n(a, b, c)$.

Case 2: $2i < n$.
Let uv be an edge of the unique cycle C_l of G. By Lemma 2.1 we have

$$a_{2i}(G^\sigma) = m(G, i) = m(G - uv, i) + m(G - u - v, i - 1),$$

$$a_{2i}(C_n^+) = m(C_n, i) = m(P_n, i) + m(P_{n-2}, i - 1).$$

Since $G-uv$ and $G-u-v$ are acyclic, we have $m(G-uv, i) \leqslant m(P_n, i)$ and $m(G-u-v, i-1) \leqslant m(P_{n-2}, i - 1)$. Then $a_{2i}(G^\sigma) \leqslant a_{2i}(C_n^+)$. This implies that $G^\sigma \prec C_n^+$.

(2) The proof is similar to (1). □

Let $P_n(a, b, c)$ be a tree of order n obtained by attaching three pendent paths of length a, b and c to an isolated vertex u where $a + b + c = n - 1$ (see Fig. 4).

Lemma 2.3. *Let $n \geqslant 10$. Then we have:*

(1) *If n is even, then $C_n^+ \prec F_n^+$;*
(2) *If n is odd, then $C_n^* \prec F_n^+$.*

Proof. (1) By Lemma 1.2 we have $a_{2i}(F_n^+) = m(F_n, i) + 2m(P_{n-4}(2, 2, n-9), i-2)$. We consider the following two cases:

Case 1: $2i = n$.
Then $a_n(F_n^+) = m\left(F_n, \frac{n}{2}\right) + 2 = 4$ and $a_n(C_n^+) = m\left(C_n, \frac{n}{2}\right) + 2 = 4$.

Case 2: $2i < n$.
Let u_3u_4 be an edge of the unique cycle C_4 of F_n (see Fig. 3). By Lemma 2.1 we have

$$\begin{aligned} m(F_n, i) &= m(F_n - u_3u_4, i) + m(F_n - u_3 - u_4, i - 1) \\ &= m(P_n(2, 2, n-5), i) + m\left(P_2 \cup P_{n-4}(2, 2, n-9), i - 1\right). \end{aligned}$$

Furthermore, by Lemma 2.1 we have

$$m(P_n(2, 2, n-5), i) = m\left(P_2 \cup P_{n-2}, i\right) + m\left(P_2 \cup P_{n-5}, i - 1\right),$$
$$m\left(P_2 \cup P_{n-4}(2, 2, n-9), i - 1\right) = m\left(P_2 \cup P_2 \cup P_{n-6}, i - 1\right) + m\left(P_2 \cup P_2 \cup P_{n-9}, i - 2\right).$$

By Lemma 1.2, we have $a_{2i}(C_n^+) = m(C_n, i) = m(P_n, i) + m(P_{n-2}, i-1)$. Moreover, by Lemma 2.1 we have

$$\begin{aligned} m(P_n, i) &= m\,(P_2 \cup P_{n-2}, i) + m(P_{n-3}, i-1) \\ &= m\,(P_2 \cup P_{n-2}, i) + m\,(P_2 \cup P_{n-5}, i-1) + m(P_{n-6}, i-2), \end{aligned}$$

$$\begin{aligned} m(P_{n-2}, i-1) &= m\,(P_2 \cup P_{n-4}, i-1) + m(P_{n-5}, i-2) \\ &= m\,(P_2 \cup P_2 \cup P_{n-6}, i-1) + m(P_2 \cup P_{n-7}, i-2) + m(P_{n-5}, i-2). \end{aligned}$$

It follows that

$$a_{2i}(F_n^+) - a_{2i}(C_n^+) = 2m(P_{n-4}(2, 2, n-9), i-2) + m(P_2 \cup P_2 \cup P_{n-9}, i-2)$$

$$-m(P_{n-6}, i-2) - m(P_2 \cup P_{n-7}, i-2) - m(P_{n-5}, i-2).$$

Next,

$$\begin{aligned} m(P_{n-4}(2, 2, n-9), i-2) &= m(P_2 \cup P_{n-6}, i-2) + m(P_2 \cup P_{n-9}, i-3) \\ &= m(P_{n-6}, i-2) + m(P_{n-6}, i-3) + m(P_2 \cup P_{n-9}, i-3), \end{aligned}$$

$$m(P_2 \cup P_{n-7}, i-2) - m(P_2 \cup P_2 \cup P_{n-9}, i-2) = m(P_2 \cup P_{n-10}, i-3).$$

By Lemma 2.1 we have

$$\begin{aligned} a_{2i}(F_n^+) - a_{2i}(C_n^+) &\geqslant m(P_{n-6}, i-2) + 2m(P_{n-6}, i-3) - m(P_{n-5}, i-2) \\ &= 2m(P_{n-6}, i-3) - m(P_{n-7}, i-3) \\ &\geqslant m(P_{n-6}, i-3) \\ &\geqslant 0. \end{aligned}$$

Then $a_{2i}(F_n^+) \geqslant a_{2i}(C_n^+)$. Moreover, $a_6(F_n^+) - a_6(C_n^+) \geqslant 1 > 0$. This implies that $C_n^+ \prec F_n^+$.

(2) The proof is similar to (1). □

3. *l* is even

Let $G \in G(n, l)$ and C_l be the unique cycle of G. For convenience, we always assume that l is even in this section. The main result in this section is that if $G^\sigma \neq U_4^+(a, b)$ with $a + b = n - 5$, then $G^\sigma \preceq F_n^+$.

First, when C_l is evenly oriented, we can easily obtain the following lemma.

Lemma 3.1. *Let $G \in G(n, l)$ and l be even. Then $G^- \prec F_n^+$.*

Proof. By Lemma 1.2, we have that $a_{2i}(G^-) = m(G, i) - 2m(G - C_l, i - \frac{l}{2}) \leqslant m(G, i)$. Using the same method as in Lemma 2.2, we can show $G^- \preceq C_n^*$ or $G^- \preceq C_n^+$. By Lemma 2.3, the result holds. □

Next, we need to consider the case that C_l is oddly oriented. In order to prove the main result, we take two steps to consider the problem: $l = 4$ and $l \geqslant 6$.

We first consider the case $l = 4$. It is easy to see that each unicyclic graph can be obtained by attaching rooted trees to the vertices of a cycle C_l. Let $G \in G(n, 4)$ and $C_4 = x_1x_2x_3x_4x_1$ be the unique cycle of G. Let $T_{a+1}, T_{b+1}, T_{c+1}, T_{d+1}$ be four rooted trees of order $a+1, b+1, c+1, d+1$, respectively. We denote by $C_4(T_a, T_b, T_c, T_d)$ the graph obtained by attaching four rooted trees $T_{a+1}, T_{b+1}, T_{c+1}, T_{d+1}$

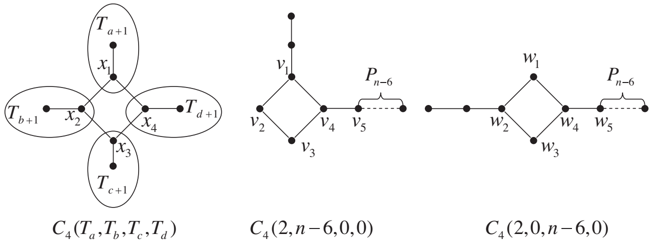

Fig. 5. Graphs $C_4(T_a, T_b, T_c, T_d)$, $C_4(2, n-6, 0, 0)$ and $C_4(2, 0, n-6, 0)$.

to the vertex x_i (i=1,2,3,4) respectively. In particular, when $T_{a+1}, T_{b+1}, T_{c+1}, T_{d+1}$ are four paths with their endpoints as their roots, we write $C_4(a, b, c, d)$ as $C_4(T_a, T_b, T_c, T_d)$ (see Fig. 5).

In [25], Gutman shown that the first and second maximal energy trees of order n are P_n and $P_n(2, 2, n-5)$. The following lemma is an alternative form of Proposition 9 in [25].

Lemma 3.2 [25]. *Let T be a tree of order n. If $T \neq P_n$, then $m(T, i) \leqslant m(P_n(2, 2, n-5), i)$.*

The following two lemmas only are used in Theorem 3.1 to prove that if $C_4(T_a, T_b, T_c, T_d) \neq U_4(s, t)$ with $a+b+c+d = n-4$ and $s+t = n-5$, then $C_4^+(T_a, T_b, T_c, T_d) \preceq F_n^+$.

Lemma 3.3. *Let $n \geqslant 10$. Then $C_4^+(2, 0, n-6, 0) \preceq C_4^+(2, n-6, 0, 0)$.*

Proof. By Lemma 1.2 we have

$$\begin{aligned} a_{2i}(C_4^+(2, n-6, 0, 0)) &= m(C_4(2, n-6, 0, 0), i) + 2m(P_2 \cup P_{n-6}, i-2), \\ a_{2i}(C_4^+(2, 0, n-6, 0)) &= m(C_4(2, 0, n-6, 0), i) + 2m(P_2 \cup P_{n-6}, i-2). \end{aligned}$$

Let v_3v_4 be an edge of C_4 of $C_4(2, n-6, 0, 0)$ and w_3w_4 be an edge of C_4 of $C_4(2, 0, n-6, 0)$ (see Fig. 5). By Lemma 2.1, we have

$$\begin{aligned} & m(C_4(2, n-6, 0, 0), i) \\ &\quad = m(C_4(2, n-6, 0, 0) - v_3v_4, i) + m(C_4(2, n-6, 0, 0) - v_3 - v_4, i-1) \\ &\quad = m(P_n(2, 2, n-5), i) + m(P_4 \cup P_{n-6}, i-1), \\ & m(C_4(2, 0, n-6, 0), i) \\ &\quad = m(C_4(2, 0, n-6, 0) - w_3w_4, i) + m(C_4(2, 0, n-6, 0) - w_3 - w_4, i-1) \\ &\quad = m(P_n(1, 2, n-4), i) + m(P_4 \cup P_{n-6}, i-1). \end{aligned}$$

By Lemma 3.2 we have $m(P_n(2, 2, n-5), i) \geqslant m(P_n(1, 2, n-4), i)$. It follows that $a_{2i}(C_4^+(2, n-6, 0, 0)) \geqslant a_{2i}(C_4^+(2, 0, n-6, 0))$. It implies that $C_4^+(2, 0, n-6, 0) \preceq C_4^+(2, n-6, 0, 0)$. $\square$

Lemma 3.4. *Let $n \geqslant 10$. Then $C_4^+(2, n-6, 0, 0) \prec F_n^+$.*

Proof. By Lemma 1.2, we have

$$a_{2i}(F_n^+) = m(F_n, i) + 2m(P_{n-4}(2, 2, n-9), i-2),$$

$$a_{2i}(C_4^+(2, n-6, 0, 0)) = m(C_4(2, n-6, 0, 0), i) + 2m(P_2 \cup P_{n-6}, i-2).$$

Let u_3u_4 be an edge of the unique cycle C_4 of F_n and v_3v_4 be an edge of the unique cycle C_4 of $C_4(2, n-6, 0, 0)$ (see Figs. 3 and 5). By Lemma 2.1 we have

$$\begin{aligned} m(F_n, i) &= m(F_n - u_3u_4, i) + m(F_n - u_3 - u_4, i-1) \\ &= m(P_n(2, 2, n-5), i) + m(P_2 \cup P_{n-4}(2, 2, n-9), i-1), \end{aligned}$$

$$\begin{aligned} m(C_4(2, n-6, 0, 0), i) &= m(C_4(2, n-6, 0, 0) - v_3v_4, i) \\ &\quad + m(C_4(2, n-6, 0, 0) - v_3 - v_4, i-1) \\ &= m(P_n(2, 2, n-5), i) + m(P_4 \cup P_{n-6}, i-1), \end{aligned}$$

$$m(P_2 \cup P_{n-4}(2, 2, n-9), i-1) = m(P_2 \cup P_2 \cup P_{n-6}, i-1) + m(P_2 \cup P_2 \cup P_{n-9}, i-2),$$

$$m(P_4 \cup P_{n-6}, i-1) = m(P_2 \cup P_2 \cup P_{n-6}, i-1) + m(P_{n-6}, i-2).$$

Thus, $m(F_n, i) - m(C_4(2, n-6, 0, 0), i) = m(P_2 \cup P_2 \cup P_{n-9}, i-2) - m(P_{n-6}, i-2)$. By Lemma 2.1 we have

$$m(P_{n-4}(2, 2, n-9), i-2) = m(P_2 \cup P_{n-6}, i-2) + m(P_2 \cup P_{n-9}, i-3).$$

It follows that

$$a_{2i}(F_n^+) - a_{2i}(C_4^+(2, n-6, 0, 0)) = 2m(P_2 \cup P_{n-9}, i-3) + m(P_2 \cup P_2 \cup P_{n-9}, i-2) - m(P_{n-6}, i-2)$$

Next,

$$m(P_{n-6}, i-2) = m(P_3 \cup P_{n-9}, i-2) + m(P_2 \cup P_{n-10}, i-3),$$

$$m(P_2 \cup P_2 \cup P_{n-9}, i-2) \geqslant m(P_3 \cup P_{n-9}, i-2),$$

$$m(P_2 \cup P_{n-9}, i-3) \geqslant m(P_2 \cup P_{n-10}, i-3).$$

Then $a_{2i}(F_n^+) - a_{2i}(C_4^+(2, n-6, 0, 0)) \geqslant m(P_2 \cup P_{n-9}, i-3) \geqslant 0$. Moreover, $a_6(F_n^+) - a_6(C_4^+(2, n-6, 0, 0)) \geqslant 1 > 0$. It implies that $C_4^+(2, n-6, 0, 0) \prec F_n^+$. $\square$

The following lemma is an alternative form of Theorem 2.2 in [26] which will be used to compare the matching numbers of two trees in Theorem 3.1 and Lemmas 3.8, 3.9.

Lemma 3.5 [26]. *Let $a + b = c + d$ with $0 \leqslant a \leqslant b$ and $0 \leqslant c \leqslant d$. Let $a < c$. Then we have:*

(1) *If a is even, then $m(P_a \cup P_b, i) \geqslant m(P_c \cup P_d, i)$. Furthermore, there exists at least one index i such that the above inequality is strict.*
(2) *If a is odd, then $m(P_a \cup P_b, i) \leqslant m(P_c \cup P_d, i)$. Furthermore, there exists at least one index i such that the above inequality is strict.*

Let a, b be nonnegative integers and $\mathcal{B}_n = \{U_4(a, b) | 0 \leqslant a \leqslant b,\ a + b = n - 5\}$. We denote by $d_G(u)$ the degree of vertex u in graph G.

Theorem 3.1. *Let $G \in G(n, 4)$ and $n \geqslant 10$. If $G \notin \mathcal{B}_n$, then $G^+ \preceq F_n^+$.*

Proof. Let $C_4 = x_1x_2x_3x_4x_1$ be the unique cycle of G. Let $N(G) = \{x_i | d_G(x_i) \geqslant 3,\ i = 1, 2, 3, 4\}$. We consider the following four cases:

Case 1: $|N(G)| = 1$.

Without loss of generality, we assume $d_G(x_1) \geqslant 3$. Then G must be the form of $C_4(T_{n-4}, 0, 0, 0)$. By Lemma 1.2 we have

$$a_{2i}(C_4^+(T_{n-4}, 0, 0, 0)) = m(C_4(T_{n-4}, 0, 0, 0), i) + 2m(C_4(T_{n-4}, 0, 0, 0) - C_4, i-2),$$

$$a_{2i}(F_n^+) = m(F_n, i) + 2m(P_{n-4}(2, 2, n-9), i-2).$$

Since $G \notin \mathcal{B}_n$, we have $C_4(T_{n-4}, 0, 0, 0) - C_4 \neq P_{n-4}$. By Lemma 3.2 we have $m(C_4(T_{n-4}, 0, 0, 0) - C_4, i-2) \leqslant m(P_{n-4}(2, 2, n-9), i-2)$.

Let u_3u_4 be an edge of the unique cycle C_4 of F_n (see Fig. 3). By Lemma 2.1, we have

$$\begin{aligned} m(F_n, i) &= m(F_n - u_3u_4, i) + m(F_n - u_3 - u_4, i-1) \\ &= m(P_n(2, 2, n-5), i) + m(P_2 \cup P_{n-4}(2, 2, n-9), i-1), \end{aligned}$$

$$m(C_4(T_{n-4}, 0, 0, 0), i) = m(C_4(T_{n-4}, 0, 0, 0) - x_1x_2, i) + m(C_4(T_{n-4}, 0, 0, 0) - x_1 - x_2, i-1).$$

By Lemma 3.2, we have

$$m(C_4(T_{n-4}, 0, 0, 0) - x_1x_2, i) \leqslant m(P_n(2, 2, n-5), i),$$

$$m(C_4(T_{n-4}, 0, 0, 0) - x_1 - x_2, i-1) \leqslant m(P_2 \cup P_{n-4}(2, 2, n-9), i-1).$$

It follows that $m(C_4(T_{n-4}, 0, 0, 0), i) \leqslant m(F_n, i)$. Then $a_{2i}(C_4^+(T_{n-4}, 0, 0, 0)) \leqslant a_{2i}(F_n^+)$. It implies that $C_4^+(T_{n-4}, 0, 0, 0) \preceq F_n^+$.

Case 2: $|N(G)| = 2$.

Then G must be the form of $C_4(T_a, T_b, 0, 0)$ or $C_4(T_a, 0, T_b, 0)$ with $a + b = n - 4$.

Subcase 2.1: G is the form of $C_4(T_a, T_b, 0, 0)$.

Without loss of generality, we assume $d_G(x_1) \geqslant 3$ and $d_G(x_2) \geqslant 3$. By Lemma 1.2, we have

$$\begin{aligned} a_{2i}(C_4^+(T_a, T_b, 0, 0)) &= m(C_4(T_a, T_b, 0, 0), i) + 2m(C_4(T_a, T_b, 0, 0) - C_4, i-2) \\ &\leqslant m(C_4(T_a, T_b, 0, 0), i) + 2m(T_a \cup T_b, i-2), \\ a_{2i}(C_4^+(2, n-6, 0, 0)) &= m(C_4(2, n-6, 0, 0), i) + 2m(P_2 \cup P_{n-6}, i-2). \end{aligned}$$

Let v_1v_4 be an edge of C_4 of $C_4(2, n-6, 0, 0)$ (see Fig. 5). By Lemma 2.1, we have

$$\begin{aligned} m(C_4(2, n-6, 0, 0), i) &= m(C_4(2, n-6, 0, 0) - v_1v_4, i) \\ &\quad + m(C_4(2, n-6, 0, 0) - v_1 - v_4, i-1) \\ &= m(P_n, i) + m(P_2 \cup P_2 \cup P_{n-6}, i-1), \\ m(C_4(T_a, T_b, 0, 0), i) &= m(C_4(T_a, T_b, 0, 0) - x_1x_2, i) \\ &\quad + m(C_4(T_a, T_b, 0, 0) - x_1 - x_2, i-1) \\ &\leqslant m(P_n, i) + m(P_2 \cup T_a \cup T_b, i-1). \end{aligned}$$

Then $m(C_4(T_a, T_b, 0, 0) - x_1x_2, i) \leqslant m(P_n, i)$. Since $T_a \cup T_b$ is a unconnected forest of order $n-4$, by Lemma 3.5 we have

$$\begin{aligned} m(T_a \cup T_b, i-2) &\leqslant m(P_2 \cup P_{n-6}, i-2), \\ m(P_2 \cup T_a \cup T_b, i-1) &\leqslant m(P_2 \cup P_2 \cup P_{n-6}, i-1). \end{aligned}$$

It follows that $a_{2i}(C_4^+(T_a, T_b, 0, 0)) \leqslant a_{2i}(C_4^+(2, n-6, 0, 0))$. Then $C_4^+(T_a, T_b, 0, 0) \preceq C_4^+(2, n-6, 0, 0)$. By Lemma 3.4, we have $C_4^+(T_a, T_b, 0, 0) \prec F_n^+$.

Subcase 2.2: G is the form of $C_4(T_a, 0, T_b, 0)$.

Using the same method as in Subcase 2.1, we can obtain that $C_4^+(T_a, 0, T_b, 0) \preceq C_4^+(2, 0, n-6, 0)$. By Lemmas 3.3 and 3.4 we have $C_4^+(T_a, 0, T_b, 0) \prec F_n^+$.

Case 3: $|N(G)| = 3$.

Then G must be the form of $C_4(T_a, T_b, T_c, 0)$ with $a+b+c = n-4$. Without loss of generality, we assume $d_G(x_1) \geqslant 3$, $d_G(x_2) \geqslant 3$ and $d_G(x_3) \geqslant 3$. By Lemmas 1.2, 2.1, 3.5, we have

$$\begin{aligned} &a_{2i}(C_4^+(T_a, T_b, T_c, 0)) \\ &\quad = m(C_4(T_a, T_b, T_c, 0), i) + 2m(C_4(T_a, T_b, T_c, 0) - C_4, i-2) \\ &\quad \leqslant m(C_4(T_a, T_b, T_c, 0) - x_1x_2, i) + m(C_4(T_a, T_b, T_c, 0) - x_1 - x_2, i-1) \\ &\quad +2m(T_a \cup T_b \cup T_c, i-2) \\ &\quad \leqslant m(P_n, i) + m(P_2 \cup P_2 \cup P_{n-6}, i-1) + 2m(P_2 \cup P_{n-6}, i-2). \end{aligned}$$

According to the computing in Subcase 2.1, we have

$$a_{2i}(C_4^+(2, n-6, 0, 0)) = m(P_n, i) + m(P_2 \cup P_2 \cup P_{n-6}, i-1) + 2m(P_2 \cup P_{n-6}, i-2).$$

Then $a_{2i}(C_4^+(T_a, T_b, T_c, 0)) \preceq a_{2i}(C_4^+(2, n-6, 0, 0))$. It implies that $C_4^+(T_a, T_b, T_c, 0) \preceq C_4^+(2, n-6, 0, 0)$. By Lemma 3.4 we have $C_4^+(T_a, T_b, T_c, 0) \prec F_n^+$.

Case 4: $|N(G)| = 4$.

Then G must be the form of $C_4(T_a, T_b, T_c, T_d)$ with $a+b+c+d = n-4$. By Lemmas 1.2, 2.1, 3.5, we have

$$\begin{aligned} a_{2i}(C_4^+(T_a, T_b, T_c, T_d)) &\leqslant m(C_4(T_a, T_b, T_c, T_d), i) + 2m(T_a \cup T_b \cup T_c \cup T_d, i-2) \\ &= m(C_4(T_a, T_b, T_c, T_d) - x_1x_2, i) + m(C_4(T_a, T_b, T_c, T_d) \\ &\quad -x_1 - x_2, i-1) + 2m(T_a \cup T_b \cup T_c \cup T_d, i-2) \\ &\leqslant m(P_n, i) + m(P_2 \cup P_2 \cup P_{n-6}, i-1) + 2m(P_2 \cup P_{n-6}, i-2). \end{aligned}$$

Then $a_{2i}(C_4^+(T_a, T_b, T_c, T_d)) \preceq a_{2i}(C_4^+(2, n-6, 0, 0))$. It follows that $C_4^+(T_a, T_b, T_c, T_d) \preceq C_4^+(2, n-6, 0, 0)$. By Lemma 3.4 we have $C_4^+(T_a, T_b, T_c, T_d) \prec F_n^+$.

To conclude, the result holds. □

Next, we are ready to consider the case that $l \geqslant 6$. First, we will prove that if $G \neq P_n^{n-2}$, then $G^+ \preceq (P_n^6)^+$. Then we will show that $(P_n^{n-2})^+ \prec F_n^+$ and $(P_n^6)^+ \prec F_n^+$.

In [10], Hou et al. proved the following result by induction on $n-l$.

Lemma 3.6 [10]. *Let $G \in G(n, l)$. If $G \neq P_n^l$, then $G^+ \prec (P_n^l)^+$.*

The following lemma about the matching number will be used in Lemmas 3.8 and 3.9 to study the skew energies of the oriented unicyclic graphs $(P_n^l)^+$.

Lemma 3.7. *Let k, r, i be nonnegative integers. If $k \geqslant 2r+1$ and $i \geqslant r$, then $m(P_{k-2r}, i-r) \leqslant m(P_k, i)$.*

Proof. By Lemma 2.1 we have

$$\begin{aligned} m(P_k, i) &= m(P_{2r} \cup P_{k-2r}, i) + m(P_{2r-1} \cup P_{k-2r-1}, i-1) \\ &= \sum_{t=0}^{i} [m(P_{2r}, t) \cdot m(P_{k-2r}, i-t)] + m(P_{2r-1} \cup P_{k-2r-1}, i-1) \\ &\geqslant m(P_{k-2r}, i-r) + m(P_{2r-1} \cup P_{k-2r-1}, i-1) \\ &\geqslant m(P_{k-2r}, i-r). \quad \square \end{aligned}$$

When n and l ($l \geqslant 6$) are even, the following two lemmas show that if $P_n^l \neq P_n^{n-2}$, then $(P_n^l)^+ \preceq (P_n^6)^+$.

Lemma 3.8. *Let n and l be even. Then we have:*

(1) *If $6 \leqslant l \leqslant \frac{n}{2}$, then $(P_n^l)^+ \succ (P_n^{l+2})^+$.*
(2) *If $6 \leqslant l < \frac{n}{2} + 1$, then $(P_n^l)^+ \succ (P_n^{n-l+2})^+$.*

Proof.

(1) By Lemmas 1.2 and 2.1, we have

$$\begin{aligned} a_{2i}((P_n^l)^+) &= m(P_n^l, i) + 2m\left(P_{n-l}, i - \frac{l}{2}\right) \\ &= m(P_n, i) + m(P_{l-2} \cup P_{n-l}, i-1) + 2m\left(P_{n-l}, i - \frac{l}{2}\right), \\ a_{2i}((P_n^{l+2})^+) &= m(P_n^{l+2}, i) + 2m\left(P_{n-l-2}, i - \frac{l+2}{2}\right) \\ &= m(P_n, i) + m(P_l \cup P_{n-l-2}, i-1) + 2m\left(P_{n-l-2}, i - \frac{l+2}{2}\right). \end{aligned}$$

By Lemma 3.7 we have $m\left(P_{n-l}, i - \frac{l}{2}\right) \geqslant m\left(P_{n-l-2}, i - \frac{l+2}{2}\right)$.
Since $l \leqslant \frac{n}{2}$, we have $l - 2 \leqslant n - l - 2$. Since $l - 2$ is even, by Lemma 3.5 (1) we have $m(P_{l-2} \cup P_{n-l}, i-1) \geqslant m(P_l \cup P_{n-l-2}, i-1)$. Then $a_{2i}((P_n^l)^+) \geqslant a_{2i}((P_n^{l+2})^+)$.
When $i = \frac{l+2}{2}$, we have

$$m\left(P_{n-l}, i - \frac{l}{2}\right) = m(P_{n-l}, 1) = n - l > m\left(P_{n-l-2}, i - \frac{l+2}{2}\right) = m(P_{n-l-2}, 0) = 1.$$

Thus $a_{l+2}\left((P_n^l)^+\right) > a_{l+2}\left((P_n^{l+2})^+\right)$. It follows that $(P_n^l)^+ \succ \left(P_n^{l+2}\right)^+$.
(2) The proof is similar to (1). $\square$

Lemma 3.9. *Let n be odd and $6 \leqslant l \leqslant n-3$. If l is even, then $(P_n^l)^+ \succ \left(P_n^{l+2}\right)^+$.*

Proof. By Lemmas 1.2 and 2.1, we have

$$\begin{aligned} a_{2i}((P_n^l)^+) &= m(P_n^l, i) + 2m\left(P_{n-l}, i - \frac{l}{2}\right) \\ &= m(P_n, i) + m(P_{l-2} \cup P_{n-l}, i-1) + 2m\left(P_{n-l}, i - \frac{l}{2}\right), \\ a_{2i}((P_n^{l+2})^+) &= m(P_n^{l+2}, i) + 2m\left(P_{n-l-2}, i - \frac{l+2}{2}\right) \\ &= m(P_n, i) + m(P_l \cup P_{n-l-2}, i-1) + 2m\left(P_{n-l-2}, i - \frac{l+2}{2}\right). \end{aligned}$$

By Lemma 3.7 we have $m\left(P_{n-l}, i - \frac{l}{2}\right) \geqslant m(P_{n-l-2}, i - \frac{l+2}{2})$. We consider the following two cases:

Case 1: $l-2 \leqslant n-l-2$.
Since $l-2$ is even, by Lemma 3.5 (1), we have

$$m(P_{l-2} \cup P_{n-l}, i-1) \geqslant m(P_l \cup P_{n-l-2}, i-1).$$

Thus $a_{2i}((P_n^l)^+) \geqslant a_{2i}((P_n^{l+2})^+)$.

Case 2: $l-2 > n-l-2$.
Since $n-l-2$ is odd, by Lemma 3.5 (2), we have

$$m(P_{l-2} \cup P_{n-l}, i-1) \geqslant m(P_l \cup P_{n-l-2}, i-1).$$

Thus $a_{2i}((P_n^l)^+) \geqslant a_{2i}((P_n^{l+2})^+)$.
When $i = \frac{l+2}{2}$, we have

$$m\left(P_{n-l}, i - \frac{l}{2}\right) = m(P_{n-l}, 1) = n-l > m\left(P_{n-l-2}, i - \frac{l+2}{2}\right) = m(P_{n-l-2}, 0) = 1.$$

Thus $a_{l+2}((P_n^l)^+) > a_{l+2}((P_n^{l+2})^+)$. Consequently, $(P_n^l)^+ \succ (P_n^{l+2})^+$. $\square$

By Lemmas 3.6, 3.8 and 3.9, we can obtain that if $G \neq P_n^{n-2}, P_n^4$, then $G^+ \preceq (P_n^6)^+$. In order to prove the main result in this section, we now only need to prove that $(P_n^{n-2})^+ \prec F_n^+$ and $(P_n^6)^+ \prec F_n^+$.

Lemma 3.10. *If $n \geqslant 10$, then $(P_n^{n-2})^+ \prec F_n^+$.*

Proof. First we prove that $(P_n^{n-2})^+ \prec C_n^+$. By Lemma 1.2 we have $a_n((P_n^{n-2})^+) = a_n(C_n^+) = 4$.
When $2i < n$, by Lemmas 1.2 and 2.1 we have

$$\begin{aligned} a_{2i}((P_n^{n-2})^+) &= m(P_n^{n-2}, i) + 2m(P_2, i - \tfrac{n-2}{2}) \\ &= m(P_n, i) + m(P_2 \cup P_{n-4}, i-1) + 2m(P_2, i - \tfrac{n-2}{2}), \end{aligned}$$

$$a_{2i}(C_n^+) = m(C_n, i) = m(P_n, i) + m(P_{n-2}, i-1).$$

We consider the following two cases:

Case 1: $2i = n-2$.

Then

$$\begin{aligned} a_{n-2}((P_n^{n-2})^+) &= m\left(P_n, \tfrac{n-2}{2}\right) + m\left(P_2 \cup P_{n-4}, \tfrac{n-4}{2}\right) + 2, \\ a_{n-2}(C_n^+) &= m(C_n, , \tfrac{n-2}{2}) = m\left(P_n, \tfrac{n-2}{2}\right) + m\left(P_{n-2}, , \tfrac{n-4}{2}\right). \end{aligned}$$

Thus,

$$\begin{aligned} a_{n-2}(C_n^+) - a_{n-2}((P_n^{n-2})^+) &= m\left(P_{n-2}, , \tfrac{n-4}{2}\right) - m\left(P_2 \cup P_{n-4}, \tfrac{n-4}{2}\right) - 2 \\ &= m\left(P_{n-5}, \tfrac{n-6}{2}\right) - 2 \\ &= 1 + m\left(P_{n-7}, \tfrac{n-8}{2}\right) - 2 \\ &= m\left(P_{n-9}, \tfrac{n-10}{2}\right) \\ &> 0. \end{aligned}$$

Case 2: $2i < n - 2$.

$$\begin{aligned} a_{2i}((P_n^{n-2})^+) &= m(P_n, i) + m(P_2 \cup P_{n-4}, i-1), \\ a_{2i}(C_n^+) = m(C_n, i) &= m(P_n, i) + m(P_{n-2}, i-1). \end{aligned}$$

By Lemma 3.5 (1), we have $a_{2i}((P_n^{n-2})^+) \leqslant a_{2i}(C_n^+)$. Then $(P_n^{n-2})^+ \prec C_n^+$. By Lemma 2.3 (1), we have $(P_n^{n-2})^+ \prec F_n^+$. $\square$

Lemma 3.11. *If $n \geqslant 10$, then $(P_n^6)^+ \prec F_n^+$.*

Proof. By Lemma 1.2 we have

$$\begin{aligned} a_{2i}((P_n^6)^+) &= m(P_n^6, i) + 2m(P_{n-6}, i-3), \\ a_{2i}(F_n^+) &= m(F_n, i) + 2m(P_{n-4}(2, 2, n-9), i-2). \end{aligned}$$

Let u_3u_4 be an edge of the unique cycle C_4 of F_n (see Fig. 3). By Lemma 2.1 we have

$$\begin{aligned} m(F_n, i) &= m(F_n - u_3u_4, i) + m(F_n - u_3 - u_4, i-1) \\ &= m(P_n(2, 2, n-5), i) + m(P_2 \cup P_{n-4}(2, 2, n-9), i-1). \\ m(P_n^6, i) &= m(P_n, i) + m(P_4 \cup P_{n-6}, i-1). \end{aligned}$$

Furthermore, by Lemma 2.1 we have

$$\begin{aligned} m(P_n(2, 2, n-5), i) &= m(P_2 \cup P_{n-2}, i) + m(P_2 \cup P_{n-5}, i-1), \\ m(P_n, i) &= m(P_2 \cup P_{n-2}, i) + m(P_{n-3}, i-1) \\ &= m(P_2 \cup P_{n-2}, i) + m(P_2 \cup P_{n-5}, i-1) + m(P_{n-6}, i-2), \\ m(P_2 \cup P_{n-4}(2, 2, n-9), i-1) &= m(P_2 \cup P_2 \cup P_{n-6}, i-1) + m(P_2 \cup P_2 \cup P_{n-9}, i-2), \\ m(P_4 \cup P_{n-6}, i-1) &= m(P_2 \cup P_2 \cup P_{n-6}, i-1) + m(P_{n-6}, i-2). \end{aligned}$$

Then $m(F_n, i) - m(P_n^6, i) = m(P_2 \cup P_2 \cup P_{n-9}, i-2) - 2m(P_{n-6}, i-2)$. By Lemma 2.1 we have

$$\begin{aligned} m(P_{n-4}(2, 2, n-9), i-2) &= m(P_2 \cup P_{n-6}, i-2) + m(P_2 \cup P_{n-9}, i-3) \\ &= m(P_{n-6}, i-2) + m(P_{n-6}, i-3) + m(P_2 \cup P_{n-9}, i-3). \end{aligned}$$

Thus,

$$a_{2i}(F_n^+) - a_{2i}((P_n^6)^+) = m(P_2 \cup P_2 \cup P_{n-9}, i-2) + 2m(P_2 \cup P_{n-9}, i-3) \geqslant 0.$$

Moreover, $a_6(F_n^+) - a_6((P_n^6)^+) = n - 6 > 0$. Then $(P_n^6)^+ \prec F_n^+$. $\square$

4. A new method of comparing the skew energies of two oriented graphs whose skew characteristic polynomials satisfy a given recurrence relation

In [27], Shan et al. presented a new method to compare the energies of two k-subdivision bipartite graphs. The main ideas can also apply to compare the skew energies of two oriented graphs whose skew characteristic polynomials satisfy a given recurrence relation.

In this section, we present a new method to compare the skew energies of two oriented graphs whose skew characteristic polynomials satisfy a given recurrence relation. As the applications of the new method, we can compare the skew energies of two oriented unicyclic graphs $U_4^+(7, n-12)$ and F_n^+ which are quasi-order incomparable.

Similar to the Coulson integral formula for the graph energy [1], Adiga et al. [6] obtained the integral formula for the skew energy of an oriented graph of order of n as follows:

Lemma 4.1 [6]. *Let $\phi(G^\sigma, x)$ be the skew characteristic polynomial of an oriented graph G^σ of order n. Then we have*

$$E_s(G^\sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[n + x\frac{\phi'(G^\sigma, -x)}{\phi(G^\sigma, -x)}\right]dx.$$

In the following two lemmas, we show the integral formula for the difference of the skew energies of two oriented graphs of order n.

Lemma 4.2.

$$\lim_{x\to\infty} x\log\frac{x^k + a_1x^{k-1} + \cdots + a_k}{x^k + b_1x^{k-1} + \cdots + b_k} = a_1 - b_1.$$

Proof. Write $t = \frac{1}{x}$.

$$\lim_{x\to\infty} x\log\frac{x^k + a_1x^{k-1} + \cdots + a_k}{x^k + b_1x^{k-1} + \cdots + b_k}$$
$$= \lim_{t\to 0}\frac{\log(1 + a_1t + \cdots + a_kt^k) - \log(1 + b_1t + \cdots + b_kt^k)}{t} = a_1 - b_1. \quad \square$$

Lemma 4.3. *Let $\phi(G_1^{\sigma_1}, x)$ and $\phi(G_2^{\sigma_2}, x)$ be two skew characteristic polynomials of two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ of order n. Then*

$$E_s(G_1^{\sigma_1}) - E_s(G_2^{\sigma_2}) = \frac{2}{\pi}\int_0^{+\infty}\log\frac{\phi(G_1^{\sigma_1}, x)}{\phi(G_2^{\sigma_2}, x)}dx.$$

Proof. For convenience, we write $\phi_1(x) = \phi(G_1^{\sigma_1}, x)$ and $\phi_2(x) = \phi(G_2^{\sigma_2}, x)$. By Lemmas 4.1 and Lemma 4.1, we have

$$\begin{aligned}
E_s(G_1^{\sigma_1}) - E_s(G_2^{\sigma_2}) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} x\left[\frac{\phi_1'(-x)}{\phi_1(-x)} - \frac{\phi_2'(-x)}{\phi_2(-x)}\right]dx \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} x\left[\frac{\phi_1'(-x)\phi_2(-x) - \phi_1(-x)\phi_2'(-x)}{\phi_1(-x)\phi_2(-x)}\right]dx \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} -x\left[\log\frac{\phi_1(-x)}{\phi_2(-x)}\right]' dx \\
&= -\frac{1}{\pi}x\log\frac{\phi_1(-x)}{\phi_2(-x)}\Big|_{-\infty}^{+\infty} + \frac{1}{\pi}\int_{-\infty}^{+\infty}\log\frac{\phi_1(-x)}{\phi_2(-x)}dx \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty}\log\frac{\phi_1(-x)}{\phi_2(-x)}dx \\
&= \frac{2}{\pi}\int_{0}^{+\infty}\log\frac{\phi_1(x)}{\phi_2(x)}dx \\
&= \frac{2}{\pi}\int_{0}^{+\infty}\log\frac{\phi(G_1^{\sigma_1},x)}{\phi(G_2^{\sigma_2},x)}dx. \qquad \square
\end{aligned}$$

In the following theorem, we present a new method to comparing the skew energies of two oriented graphs whose skew characteristic polynomials satisfy a given recurrence relation.

Theorem 4.1. *Let $f_n = \phi(G_1^{\sigma_1}, x)$ and $g_n = \phi(G_2^{\sigma_2}, x)$ be two skew characteristic polynomials of two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ of order n. Assume that $f_n = xf_{n-1} + f_{n-2}$ and $g_n = xg_{n-1} + g_{n-2}$. Write $d_n = \frac{f_n}{g_n}$. Let k_0 and k be integers with $1 \leqslant k_0 \leqslant k \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. Then we have:*

(1) *If $d_{2k_0} < d_{2k_0-1}$, then*

$$E_s(G_1^{\sigma_1}) - E_s(G_2^{\sigma_2}) = \frac{2}{\pi}\int_0^{+\infty}\log d_n dx < \frac{2}{\pi}\int_0^{+\infty}\log d_{2k-1}dx.$$

(2) *If $d_{2k_0} > d_{2k_0-1}$, then*

$$E_s(G_1^{\sigma_1}) - E_s(G_2^{\sigma_2}) = \frac{2}{\pi}\int_0^{+\infty}\log d_n dx < \frac{2}{\pi}\int_0^{+\infty}\log d_{2k}dx.$$

Proof. (1) For convenience, we assume that $x > 0$ in what follows.
First,

$$d_{2k+2} - d_{2k} = \frac{f_{2k+2}}{g_{2k+2}} - \frac{f_{2k}}{g_{2k}} = \frac{f_{2k+2}g_{2k} - f_{2k}g_{2k+2}}{g_{2k}g_{2k+2}}.$$

Since $f_{2k+2} = xf_{2k+1} + f_{2k}$ and $g_{2k+2} = xg_{2k+1} + g_{2k}$, we have

$$\begin{aligned}
f_{2k+2}g_{2k} - f_{2k}g_{2k+2} &= x(f_{2k+1}g_{2k} - f_{2k}g_{2k+1}) \\
&= x(f_{2k-1}g_{2k} - f_{2k}g_{2k-1}) \\
&= \cdots \\
&= x(f_{2k_0+1}g_{2k_0} - f_{2k_0}g_{2k_0+1}) \\
&= -x(f_{2k_0}g_{2k_0-1} - f_{2k_0-1}g_{2k_0}).
\end{aligned}$$

Then,

$$\begin{aligned} d_{2k+2} - d_{2k} &= -x\frac{f_{2k_0}g_{2k_0-1} - f_{2k_0-1}g_{2k_0}}{g_{2k}g_{2k+2}} \\ &= -x(d_{2k_0} - d_{2k_0-1})\frac{g_{2k_0-1}g_{2k_0}}{g_{2k}g_{2k+2}} \\ &> 0. \end{aligned}$$

Similarly, we have

$$d_{2k+1} - d_{2k-1} = x(d_{2k_0} - d_{2k_0-1})\frac{g_{2k_0-1}g_{2k_0}}{g_{2k-1}g_{2k+1}} < 0.$$

This implies that

$$d_{2k_0} < d_{2k_0+2} < \cdots < d_n < \cdots < d_{2k_0+1} < d_{2k_0-1}.$$

By Lemma 4.3, the result holds.

(2) The proof is similar to (1). □

Next, we use Theorem 4.1 to compare the skew energies of two oriented graphs $U_4^+(7, n-12)$ and F_n^+. Before this, we need to obtain the recurrence relation of their skew characteristic polynomials. In [10], Hou and Lei obtained the recurrence relation of the skew characteristic polynomial $\phi(G^\sigma, x)$ of an oriented graph G^σ as follows:

Lemma 4.4 [10]. *Let $e = uv$ be an edge that is on no even cycle of G and σ be an orientation of G. Then*

$$\phi(G^\sigma, x) = \phi(G^\sigma - e, x) + \phi(G^\sigma - u - v, x).$$

Let T be a tree of order n and σ_1, σ_2 be any two orientations of T. By Lemma 1.1 we have, $a_{2i}(T^{\sigma_1}) = m(T, i) = a_{2i}(T^{\sigma_2})$. Then $T^{\sigma_1} \sim T^{\sigma_2}$. For convenience, we write T^* as T^σ where σ is any orientation of T.

Lemma 4.5. *Let $n \geqslant 15$. Then we have:*

(1) $\phi(F_n^+, x) = x\phi(F_{n-1}^+, x) + \phi(F_{n-2}^+, x)$.
(2) $\phi(U_4^+(7, n-12), x) = x\phi(U_4^+(7, n-13), x) + \phi(U_4^+(7, n-14), x)$.

Proof. (1) Let u_1v_1 be a cut edge of graph F_n (see Fig. 6). By Lemma 4.4 we have

$$\phi(F_n^+, x) = \phi(F_n^+ - u_1v_1, x) + \phi(F_n^+ - u_1 - v_1, x).$$

To simplify, we let $p_n = \phi(P_n^*(2, 2, n-5), x)$. By Lemma 4.4 we have $p_n = xp_{n-1} + p_{n-2}$.

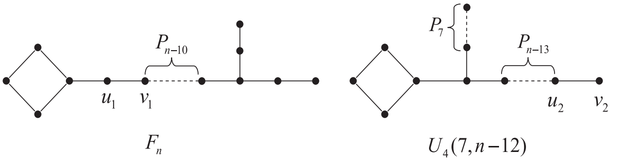

Fig. 6. Graphs F_n and $U_4(7, n-12)$.

Furthermore,

$$\phi(F_n^+ - u_1v_1, x) = \phi((P_5^4)^+, x) \cdot p_{n-5},$$
$$\phi(F_n^+ - u_1 - v_1, x) = \phi(C_4^+, x) \cdot p_{n-6}.$$

Thus,

$$\phi(F_n^+, x) = \phi((P_5^4)^+, x) \cdot p_{n-5} + \phi(C_4^+, x) \cdot p_{n-6}.$$

Then,

$$\begin{aligned}\phi(F_n^+, x) &= (\phi((P_5^4)^+, x), \phi(C_4^+, x)) \begin{pmatrix} p_{n-5} \\ p_{n-6} \end{pmatrix} \\ &= \phi((P_5^4)^+, x), \phi(C_4^+, x)) \begin{pmatrix} p_{n-6} & p_{n-7} \\ p_{n-7} & p_{n-8} \end{pmatrix} (x, 1)^T \\ &= (\phi(F_{n-1}^+, x), \phi(F_{n-2}^+, x)) \begin{pmatrix} x \\ 1 \end{pmatrix} \\ &= x\phi(F_{n-1}^+, x) + \phi(F_{n-2}^+, x).\end{aligned}$$

(2) Let u_2v_2 be a cut edge of graph $U_4(7, n-12)$ (see Fig. 6). By Lemma 4.4 we have

$$\begin{aligned}\phi(U_4^+(7, n-12), x) &= \phi(U_4^+(7, n-12) - u_2v_2, x) + \phi(U_4^+(7, n-12) - u_2 - v_2, x) \\ &= x\phi(U_4^+(7, n-13), x) + \phi(U_4^+(7, n-14), x). \quad \square\end{aligned}$$

Theorem 4.2. *Let $n \geqslant 31$. Then $E_s(F_n^+) < E_s(U_4^+(7, n-12))$.*

Proof. To simplify, we let $f_n = \phi(F_n^+, x)$ and $g_n = \phi(U_4^+(7, n-12), x)$. By Lemma 4.5 we have $f_n = xf_{n-1} + f_{n-2}$ and $g_n = xg_{n-1} + g_{n-2}$.

By some direct calculations, we have

$$\begin{aligned}f_{15} &= 26x + 181x^3 + 420x^5 + 466x^7 + 277x^9 + 90x^{11} + 15x^{13} + x^{15}, \\ g_{15} &= 24x + 180x^3 + 426x^5 + 471x^7 + 278x^9 + 90x^{11} + 15x^{13} + x^{15}, \\ f_{16} &= 4 + 96x^2 + 407x^4 + 728x^6 + 678x^8 + 354x^{10} + 104x^{12} + 16x^{14} + x^{16}, \\ g_{16} &= 4 + 96x^2 + 413x^4 + 739x^6 + 684x^8 + 355x^{10} + 104x^{12} + 16x^{14} + x^{16}.\end{aligned}$$

Furthermore,

$$f_{16}g_{15} - g_{16}f_{15} = -x(x^2+1)(x^{12} + 12x^{10} + 52x^8 + 97x^6 + 73x^4 + 22x^2 + 1)(x^2+2)^3.$$

Let $d_n = \frac{f_n}{g_n}$. If $x > 0$, then

$$d_{16} - d_{15} = \frac{f_{16}}{g_{16}} - \frac{f_{15}}{g_{15}} = \frac{f_{16}g_{15} - g_{16}f_{15}}{g_{15}g_{16}} < 0.$$

By Theorem 4.1 we have

$$E_s(F_n^+)-E_s(U_4^+(7,n-12)) = \frac{2}{\pi}\int_0^{+\infty}\log d_n dx < \frac{2}{\pi}\int_0^{+\infty}\log d_{31} dx \doteq -9.9471\times 10^{-5} < 0.$$

Then the result holds. □

5. Oriented unicyclic graphs of order *n* with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies

In this section, we can use the results in section 2, 3, 4 to determine the oriented unicyclic graphs of order $n \geqslant 31$ with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies in Theorem 5.2. First, we prove the following lemma.

Lemma 5.1. *Let $a+b=c+d$ with $0 \leqslant a \leqslant b$ and $0 \leqslant c \leqslant d$. Assume that $a < c$. Then we have:*

(1) *If a is even, then $U_4^+(a,b) \succ U_4^+(c,d)$;*
(2) *If a is odd, then $U_4^+(a,b) \prec U_4^+(c,d)$.*

Proof. By Lemma 1.2 we have

$$\begin{aligned} a_{2i}(U_4^+(a,b)) &= m(U_4(a,b),i) + 2m(P_{n-4},i-2),\\ a_{2i}(U_4^+(c,d)) &= m(U_4(c,d),i) + 2m(P_{n-4},i-2). \end{aligned}$$

Let vw be an edge of C_4 of graph $U_4(a,b)$ (see Fig. 2). By Lemma 2.1 we have

$$\begin{aligned} m(U_4(a,b),i) &= m(U_4(a,b)-vw,i) + m(U_4(a,b)-v-w,i-1)\\ &= m(P_n(4,a,b),i) + m(P_2\cup P_{n-4},i-1). \end{aligned}$$

Similarly, we have

$$m(U_4(c,d),i) = m(P_n(4,c,d),i) + m(P_2\cup P_{n-4},i-1).$$

Thus, $a_{2i}(U_4^+(a,b)) - a_{2i}(U_4^+(c,d)) = m(P_n(4,a,b),i) - m(P_n(4,c,d),i)$.
Moreover, by Lemma 2.1 we have

$$\begin{aligned} m(P_n(4,a,b),i) &= m(P_3\cup P_{n-4},i) + m(P_2\cup P_a\cup P_b,i-1),\\ m(P_n(4,c,d),i) &= m(P_3\cup P_{n-4},i) + m(P_2\cup P_c\cup P_d,i-1). \end{aligned}$$

Then $a_{2i}(U_4^+(a,b)) - a_{2i}(U_4^+(c,d)) = m(P_2\cup P_a\cup P_b,i-1) - m(P_2\cup P_c\cup P_d,i-1)$. Consequently, the results hold directly from Lemma 3.5. □

Let $\mathcal{B}_n^+ = \{U_4^+(a,b) | 0 \leqslant a \leqslant b, a+b=n-5\}$. From Lemma 5.1, we can easily obtain the following result.

Theorem 5.1. *Let $k = \left\lfloor \frac{n-5}{2} \right\rfloor$, $t = \left\lfloor \frac{k}{2} \right\rfloor$ and $l = \left\lfloor \frac{k-1}{2} \right\rfloor$. Then we have the following quasi-order relation in $\mathcal{B}_n^+$:*

$$\begin{aligned} &U_4^+(0,n-5) \succ U_4^+(2,n-7) \succ \cdots \succ U_4^+(2t,n-5-2t) \succ U_4^+(2l+1,n-5-2l-1)\\ &\succ \cdots \succ U_4^+(7,n-12) \succ U_4^+(5,n-10) \succ U_4^+(3,n-8) \succ U_4^+(1,n-6). \end{aligned}$$

Let $\mathcal{A}_n^+ = \mathcal{B}_n^+ \backslash \{U_4^+(5, n-10), U_4^+(3, n-8), U_4^+(1, n-6)\}$.

Theorem 5.2. *Let $n \geqslant 31$. The oriented unicyclic graphs of order n with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies are the oriented unicyclic graphs in $\mathcal{A}_n^+$.*

Proof. Let G^σ be an oriented unicyclic graph of order n and $G^\sigma \notin \mathcal{B}_n^+$. Let C_l be the unique cycle of G. By Theorem 5.1, we only need to prove that $E_s(G^\sigma) < E_s(U_4^+(7, n-12))$. We consider the following cases:

Case 1: l is odd.

By Lemmas 2.2 and 2.3, we have $G^\sigma \prec F_n^+$. By Theorem 4.2 we have $E_s(G^\sigma) < E_s(U_4^+(7, n-12))$.

Case 2: l is even.

By Lemma 3.1 we have $G^- \prec F_n^+$. By Theorem 4.2 we have $E_s(G^-) < E_s(U_4^+(7, n-12))$.

Subcase 2.1: $l = 4$.

By Theorem 3.1 we have $G^+ \preceq F_n^+$. By Theorem 4.2 we have $E_s(G^+) < E_s(U_4^+(7, n-12))$.

Subcase 2.2: $l \geqslant 6$.

By Lemma 3.6 we have $G^+ \preceq (P_n^l)^+$. By Lemmas 3.8 and 3.9, we have if $P_n^l \neq P_n^{n-2}$, then $(P_n^l)^+ \preceq (P_n^6)^+$.

By Lemmas 3.10 and 3.11, we have $G^+ \prec F_n^+$. By Theorem 4.2 we have $E_s(G^+) < E_s(U_4^+(7, n-12))$.

To conclude, the result holds. $\square$

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